"Particle-like" singular solutions in Einstein-Maxwell theory and in algebraic dynamics

V.V.Kassandrov[1], V.N.Trishin[2]

[1] Russian People's Friendship University

E-mail: vkassan@mx.pfu.edu.ru

[2] Moscow State University

Abstract

Foundations of *algebrodynamics* based on earlier proposed equations of *biquaternionic holomorphy* are briefly expounded. Free Maxwell and Yang-Mills Eqs. are satisfied identically on the solutions of primary system which is also related to the Eqs. of *shear-free null congruences* (SFC), and through them - to the Einstein-Maxwell electrovacuum system. *Kerr theorem* for SFC reduces the basic system to one algebraic equation, so that with each solution of the latter some (singular) solution of vacuum Eqs. may be associated. We present some exact solutions of basic algebraic and of related field Eqs. with *compact* structure of singularities of electromagnetic field, in particular having the form of *figure "8" curve*. Fundamental solution to primary system is analogous to the metric and fields of the *Kerr-Newman* solution. In addition, in the framework of algebraic dynamics the value of *electric charge* for this solution is strictly fixed in magnitude and may be set equal to the *elementary charge*.

1 . Introduction

As the main goal of the *algebrodynamical* approach [1], we regard the derivation of equations of physical fields (or of fundamental physical laws, in a broader sense) from a unique primary *Principle of purely abstract nature*, based only on the intrinsic properties of exclusive mathematical structures

1

(groups, algebras, mappings etc.). Basic realization of this concept developed in [2] makes use of only the *differentiability conditions* of functions of *biquaternionic variable*, i.e. of the Cauchy-Riemann (CR) equations generalized to the case of noncommutative (associative) algebras.

In the framework of the version of analysis proposed in [1] the noncommutativity of starting algebra naturally results in the nonlinearity of generalized CR equations (GCRE), justifying the use of the latters as the dynamical equations of *interacting* fields.

Infact, for the algebra of biquaternions $\mathbb{B}$ (isomorphic to the full 2×2 complex matrix algebra) the GCRE appear to be Lorentz invariant and carry natural 2-spinor and gauge structures. In the most examined case for which a simple geometric and physical interpretation is obvious, the conditions of $\mathbb{B}$-*differentiability* reduce to the invariant system [2]

$$\eta(x + dx) - \eta(x) \equiv d\eta = \Phi(x) * dX * \eta(x) \qquad (1)$$

for $\mathbb{C}$-valued 2-spinor $\eta(x)$ and gauge $\Phi(x)$ fields represented by 2×1 and 2×2 $\mathbb{C}$-matrices respectively. $(*)$ in (1) denotes usual matrix multiplication (equivalent to that in $\mathbb{B}$), and dX represents a 2×2 Hermitian matrix of the increments of coordinates.

As a consequence of (1) each component of the 2-spinor $\eta_A(x)(A, B = 1, 2)$ satisfy the 4-eikonal equation [6]. On the other hand, the *compatibility conditions* $d \wedge d\eta = 0$ impose dynamical restrictions on the gauge field $\Phi(x)$. Namely, the 2×2 matrix $\mathbb{C}$-valued *connection 1-form*

$$\Gamma(x) = \Phi(x) * dX = \Gamma^0(x) + \Gamma^a(x)\sigma_a \qquad (2)$$

$(\sigma_a, \; a = 1, 2, 3$ being the Pauli matrices) by virtue of compatibility conditions **should be self-dual**. Consequently, **free Maxwell and Yang-Mills equations are satistied** identically on the solutions of (1), for the scalar $\Gamma^0(x)$ and the vector $\Gamma^a(x)$ parts of the connection (2) respectively [2]. Thus, the GCRE system (1) exhibit wonderful relations to several fundamental physical equations being for the latters in some sense generating.

In the approach regarded, an important role is performed by *singular sets* where the strengths of Maxwell and YM fields turn to infinity. Such sets were found to have diverse dimensions and topology. Solutions with *compact* structure of singular set may be then considered as *particle-like*, the evolution of such singularities-particles being then governed by the system (1) itself.

In this paper, we expound the process of reduction of system (1) to the equations of *shear-free geodesic null congruences* and, by Kerr theorem, to the solution of *algebraic* equation [7]. Making use of this, we present an explicit form of the solutions to free Maxwell equations, analyze the structure

and evolution of their singular set and discuss the relation of these electromagnetic fields to the Kerr-Shild metrics and, by this, to the solutions of electrovacuum Einstein-Maxwell equations. In conclusion, we consider general status and physical interpretation of "particle-like" singular solutions in the framework of electrogravidynamics and of unified algebraic field theory proposed.

2 . Reduction of GCRE system and the solutions to free Maxwell equations.

Through the elimination of the gauge field $\Phi(x)$ the system of GCRE (1) may be written in a 2-spinor form [3]

$$\eta_{A'} \nabla^{AA'} \eta^{B'} = 0 \tag{3}$$

In the gauge $\eta^{A'}(x) = (1, G(x))$ the system (3) reduces to two equations for one unknown function $G(x)$

$$\partial_{\bar{w}} G = G \partial_u G, \qquad \partial_v G = G \partial_w G, \tag{4}$$

$u, v = t \pm z, \quad w, \bar{w} = x \pm iy$ being the *spinor coordinates*. Note that as a consequence of (4) $G(x)$ satisfy identically both the 4-eikonal and d'Alembert wave equations [3]. Assuming the Eqs. (4) are solved, the components of 4-potential matrix $\Phi(x) = A_0 + A_a(x)\sigma_a$ may be expressed through the function $G(x)$ as

$$A_w = \partial_u G, \quad A_v = \partial_w G, \quad A_u = A_{\bar{w}} = 0 \tag{5}$$

and satisfy free Maxwell equations.

Wonderfully, Eqs.(4) are completely identical to the equations of *shear-free null geodesic congruences* in the gauge for the spinor $\eta(x)$ regarded [4]. In accord with Kerr's theorem [4], we then obtain the general solution of Eqs.(4) in an implicit algebraic form

$$F(G, \bar{w}G + u, vG + w) = 0, \tag{6}$$

$F(G, \tau_1, \tau_2)$ being an arbitrary golomorphic function of three complex variables including two *twistor* (projective) components

$$\tau_A = X_{AA'} \eta^{A'}, \quad \tau_1 = \bar{w}G + u, \ \tau_2 = vG + w \tag{7}$$

Thus, a lot of solutions to free Maxwell equations may be obtained through simply examining of the algebraic Eq.(6). Singularities of related field strengths

3

may be then found from the *caustic* condition

$$\frac{dF}{dG} \equiv \partial_G F + \bar{w}\partial_{\tau_1}F + v\partial_{\tau_2}F = 0 \qquad (8)$$

Eliminating the only unknown function $G(x)$ from two algebraic Eqs.(6),(8), one easily comes to the *equation of singular set* which determines the shape and evolution of singularities, without even taking care of explicit solving the Eq.(6) itself. The example of such procedure was presented in [8].

3 . Stationary solutions.

From the structure of twistor components (7) it's evident that stationary solutions to Eq.(6) are exhausted by the functions $F(G, \lambda)$, where $\lambda = G\tau_1 - \tau_2$ doesn't contain the time variable $t = \frac{1}{2}(u + v)$ at all. In accord with the results of Kerr and Wilson [8], for stationary solutions with **compact** structure of singular set the function F **should be at most quadratic in** G, i.e. should have the form $F = (G\tau_1 - \tau_2) + a_0 G^2 + a_1 G + a_2, \quad a_i \in \mathbb{C}$. Linear dependence on G immediately leads to the trivial solution with zero fields. Using 3-translations and 3-rotations, the above form may be reduced to $F = G\tau_1 - \tau_2 - 2aG$ for which we obtain from quadratic Eq.(6)

$$G(x) = \frac{x + iy}{z - a \pm \sqrt{(z - a)^2 + x^2 + y^2}} \qquad (9)$$

For a real valued a from (9) and the expression for potentials (5) we obtain the Coulomb electric field with a point singularity and a **fixed value of the electric charge** as a consequence of nonlinear primary system of GCRE (1). Imaginary values of a correspond to the *ring singularity* with radius $r = |a|$ and multipole structure of EM fields with Coulomb first main term [7]. Decomposition and separation of real parts of EM fields at a distance $r \gg |a|$ gives

$$E_r \simeq \frac{e}{r^2}(1 - \frac{3a^2}{2r^2}(3\cos^2\theta - 1)), E_\theta \simeq -\frac{ea^2}{r^4}3\cos\theta\sin\theta,$$

$$H_r \simeq \frac{2ea}{r^3}\cos\theta, \ H_\theta \simeq \frac{ea}{r^3}\sin\theta. \qquad (10)$$

Contrary to an ambiguous value of ring's radius a, the dimensionless electric charge is strictly fixed (up to a sign) by field equations, so that in absolute units it may be identified with *elementary charge e*. Note that the solution (9) (for the case $a = 0$) and the property of charge quantization for the GCRE

4

system (1) have been obtained in a direct way in [1, 2]. (Recently [21, 22] there were some interesting attempts to explain electric charge quantization by topological reasons instead of dynamical considerations used here).

On the other hand, solution (9) is extremely important in the framework of GTR. Indeed, the expression $l_\mu = \eta^+ \sigma_\mu \eta$, where $\eta^T = (1, \; G(x))$ is the 2-spinor related to the function $G(x)$, defines the *principal null congruences* l_μ of a Riemannian space-time endowed with a Kerr-Shild metric

$$g_{\mu\nu} = \eta_{\mu\nu} + H(x) \; l_\mu l_\nu, \qquad (11)$$

where $\eta_{\mu\nu}$ represents the metric of auxiliary Minkowski space-time. The scalar factor $H(x)$ should be then determined by the Einstein vacuum or elec-trovacuum equations and for fundamental stationary solution (9) leads to the Kerr or Kerr-Newman metrics respectively (consequently, to the Schwarzschild or Reissner-Nordström metrics for the case a=0). In our approach, it is of great importance that **singularities of curvature of metric (10) are fixed [8, 9] by the same condition (8) as for electromagnetic field** and define infact one unique particle-like object. Another wonderful fact is that for the Kerr-Newman solution of Einstein-Maxwell equations electro-magnetic fields are just those defined by the GCRE system (apart from the property of quantization of charge for the latters!) and may be asymptotically presented by the Eq.(10). Moreover, these fields **obey Maxwell equations both in flat space and in Riemannian space with Kerr-Shild metric (11)!** Such a remarkable *property of stability* of electromagnetic fields under Kerr-Shild deformations of space-time geometry noticed in [10] will be discussed elsewhere.

Making use of correspondence between the fundamental solution (9) to the GCRE system (1) and the Kerr-Newman solution of Einstein-Maxwell system, one is able to endow the solution (9) with a complete set of quantum numbers (including mass and spin). Then the *gyromagnetic ratio* would au-tomatically correspond to that for Dirac particle, while the charge would be fixed in magnitude. Unfortunately, no natural reasons to ensure the quanti-zation of mass could be seen nowadays. We'll continue the discussion below.

3.1. Nonstationary solutions.

Let us consider the general quadratic form of the function $F(G, \tau_1, \tau_2)$. When the terms bilinear in τ_1, τ_2 are absent, such functions (under the restriction on singular set to be compact) correspond to the boosted or rotated Kerr solution [17]. On the other hand, the function $F = \tau_1 \tau_2 - b^2 G$ has been

considered in detail in [3]. The explicit expression for G in this case is

$$G = \frac{-2uw}{\sigma^2 + \rho^2 + b^2 \pm \sqrt{\Delta}}, \qquad \Delta \equiv (\sigma^2 + \rho^2 + b^2)^2 - 4\sigma^2\rho^2, \qquad (12)$$

where $\sigma^2 = uv = t^2 - z^2, \ \ \rho^2 = w\bar{w} = x^2 + y^2$. EM fields correspondent to (12) are

$$E_\rho = \mp\frac{8b^2\rho z}{\Delta^{3/2}}, \quad E_z = \pm\frac{4b^2}{\Delta^{3/2}}(t^2 - z^2 + \rho^2 + b^2), \quad H_\varphi = \mp\frac{8b^2\rho t}{\Delta^{3/2}}. \qquad (13)$$

For real b the fields (13) are identical to the well-known *Born solution* for two point-like charged "particles" performing uniformly accelerated counter-motion. The value of electric charge for each particle does not depend on b being **fixed and equal to the charge of fundamental solution (9)**. For the case of imaginary b one has the singularity of rather exotic toroidal structure, defined by the equation $z^2 + (\rho \pm b)^2 = t^2$ (see [3] for details). In general case of complex-valued b singular set manifests itself as the two rings of fixed radii performing again the oncoming hyperbolic motion along z-axis.

It may be proved that (up to the transformations of Poincare group) the *axisymmetric* solutions to the Eq.(6) (and to the GCRE system (1) respectively!) generated by quadratic function F **are exhausted by the Kerr-like solution (9) and the nonstationary bisingular solution (12)** together with (toroidal or double ring-like) modifications of the latter.

It seems, however, that the solutions to GCRE with compact singularity and non-axial symmetries may be of interest too. Here we present an example of such solution which may be obtained from the generating function $F = \tau_1\tau_2 - a^2G^2$. Resolving the equation $F = 0$, one comes to the following expression

$$G = \frac{2uw}{\pm\sqrt{\Delta} + uv + w\bar{w}}, \qquad (14)$$

Δ where $\Delta = (t^2 - x^2 - y^2 - z^2)^2 - 4a^2(t + z)(x + iy)$. The singular set for this solution is defined by the condition $\Delta = 0$ and for $t = 0$ has the form of flat *figure "8" curve* (Fig.a)). The time evolution of this singularity is illustrated by Fig.b).

EM fields related to the solution (14) and being represented by the complex combination $\vec{\mathcal{E}} = \vec{E} - i\vec{H}$ are:

$$\mathcal{E}_+ \equiv \mathcal{E}_1 + i\mathcal{E}_2 = -\frac{2a^2w^2}{\Delta^{3/2}}, \quad \mathcal{E}_- \equiv \mathcal{E}_1 - i\mathcal{E}_2 = \frac{2a^2u^2}{\Delta^{3/2}}, \quad \mathcal{E}_3 = -\frac{2a^2uw}{\Delta^{3/2}}. \qquad (15)$$

6

a) t=0   b) t=1
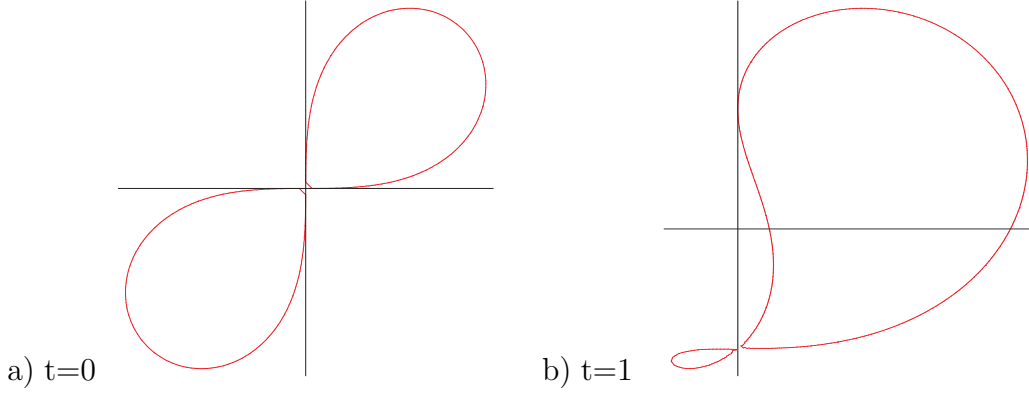
For each finite moment of time they decrease rapidly (as r^{-4}) with the distance from the centre of singularity. The fields are neutral (with total charge being equal to zero) and null ($\vec{E}^2 - \vec{H}^2 = 0, \quad \vec{E} * \vec{H} = 0$).

In conclusion, we present a peculiar solution with noncompact singularity which serves as the *analogue of electromagnetic wave* in GRCE dynamics. For the solutions of wave-like type the generating function F should depend only on one twistor component, say, τ_1. Then, for the equation $F(G, \bar{w}G + u) = 0$ the initial distribution of $G(u)$ may be arbitrary fixed at $\bar{w} = 0$, i.e. at the Z-axis. Choosing for the latter the *monochromatic* dependence and resolving the equation $G - A \exp i\Omega(\bar{w}G + u) = 0$, where the parameters A, Ω are assumed to be real and positive, we find

$$G = iW(-iA\Omega\bar{w}\exp i\Omega u)/\Omega\bar{w}, \qquad (16)$$

W being the principal branch of the so called *Lambert* function which is the solution of the equation $W(z)\exp W(z) = z$.

The structure of singular set is simply derived after that and appears to be a neutral *helix* of radius $1/\Omega Ae$ and of lead $2\pi/\Omega$ propagating along Z- direction with the speed of light. Electromagnetic fields are mutually orphogonal and transversal while polarization depends on the distance from the axis. In the direction perpendicular to the axis the fields fall at large distancies as $1/r$. As before, the fields are globally defined only up to a sign.

Shear-free null geodesic congruences l_μ and the Kerr-Shild metrics (12) may be associated with the nonstationary solutions above-presented up to the scalar factor $H(x)$. At present it's not clear if the latter may be choosed so that the Einstein-Maxwell electrovacuum system would be satisfied. By this, an interesting representation [15, 18] for the shear-free congruences (through consideration of null cone emanated by the source moving along some curve in *complex space*) as well as the *condition of stability* for electromagnetic

7

fields under the Kerr-Schild deformations of space-time geometry [10] may be of great use.

4 . General status of "particle-like" singular solutions

Well-known are the numerous problems arising in GTR and in quantum field theory in respect to the singularities of solutions of field equations (violation of causality [11, 5], divergences etc.). On the other hand, just the naked singularity of Kerr-Newman solution (which appears instead of black hole solution in the case of a large angular moment) manifests itself many remarkable properties related to that of elementary particles. Accordingly, several attempts to construct the model of electron on the base of Kerr-type solutions (KTS) have been undertaken [11, 12].

However, they all dealt with the problem of physically suitable *source* for KTS to be found which is tigthly related to the well-known *twovaluedness* of Kerr-like geometry and electromagnetic fields in particular. Infact, the introduction of source becomes admissible only after the *cut* of space which restore the global uniqueness of the $\mathbb{C}$-valued functions representing the fields of KTS.

Unfortunately, the surface of cut is quite ambigious: it may be either the *disk* spanning the Kerr singular ring [14] or the oblate spheroid [12] covering the singular ring on which the Kerr-Newman metric turns surprisingly into the Minkowski one. Consequently, one may think of the source of KTS as of the "rotating relativistic disk", of the "bubble of flat geometry" within the external Kerr-Newman space-time etc. Thus, we are to conclude that **there are no grounds to speak about the "source" of KTS solutions at all** since the twovaluedness is the unavoidable feature of their internal mathematical structure.

To illustrate the above statement, let us consider a simplier case of the singular "particle-like" solutions to free Maxwell equations in flat space-time presented in this paper. Note that all of them (apart from the Coulomb and Born solutions with point-like singularities) are of the same two-valued structure being in each point defined up to a sign. Certainly, by no δ- function distribution of charge and current along the singular curve one can reproduce the field distribution in the whole space.

On the other hand, such solutions are locally well defined and may be analytically continued from the region of regularity so that the full structure of singular set is established **in a unique way**. One cannot in any way

change either the shape and topology of the singularity or its time evolution (the latter property being the most important from physical point of view). Suppose we really hope to describe the interactions and transmutations of elementary particles by means of the solutions regarded (which are rather to be *multisingular* for real physical process). Then we'll proceed in well-defined and unique predictions in spite of partial indefinitness of EM fields and escape any divergences at all!

Moreover, one may think of such solutions as of **the only possibility to explain the "spin 1/2" structure** at a purely classical level and with transparent picture of space-time dynamics being preserved. This is still more true in the framework of the algebrodynamical approach we develop, at least for two reasons. The first one is that in respect to the internal structure of GCRE system (1), gauge (electromagnetic plus Yang-Mills) fields stand there hand by hand with the 2-spinor structure so that the latter appears naturally together with Maxwell equations. The second reason is that, apart from the right value of gyromagnetic ratio, the value of electric charge is automatically fixed by the field equations themselves.

Of course, the stationary KTS as well as bisingular and "figure 8" solutions presented here can say nothing about real dynamics of an ansamble of compact "particle-like" singularities. Even the problem of interaction of two Kerr-Newman objects is far from solution. In the framework of algebrodynamics the overdetermined structure of GCRE impose restrictions even on the initial distribution of the fields [3] so that the scattering problem should be fully reformulated.

Historically, the solution of Maxwell equations with ring singularity have been obtained by Appel in 1887 and revived in the works of Newman and Burinskii [13, 16]. General study of singular solution to Maxwell equations have been undertaken by Bateman [19]. Nowadays the concept of naked singularities of KTS as the model for elementary particles is successfully developed, say, in the works of Clement [20].

To conclude, we argue that hostile attitude of physicists to singularities of field equations could be quite unjustified. There exist no restrictions of principal character for the *compact multisingular* solutions to describe the interactions of particle-like objects in a self-consistent way. Then their transmutations could be treated as *perestroikas* of singularities in terms of catastrophe theory. This programme should be implemented independently both in the framework of Einstein-Maxwell dynamics and of the algebraic dynamics based in particular on the GCRE system (1).

References

[1] V. V. Kassandrov, "Algebraic Structure of Space-Time and Algebrodynamics", Moscow, Peop. Friend. Univ. Press., 1992 (in Russian)

[2] V. V. Kassandrov, *Grav. & Cosm.* **1**, 216, (1995)

[3] V. V. Kassandrov, J. A. Rizcalla, *in*: "Recent problems in field theory", A. V. Aminova, ed., Kazan, 1998, p.176-186 (gr-qc/9809078)

[4] R. Penrose, W. Rindler "Spinors and Space-Time", Cambridge Univ. Press, 1986

[5] R. Penrose *Riv. Nuovo Cimento* **1**, 252, (1969)

[6] V. V. Kassandrov, *Vestnik Peop. Friend. Univ., Fizika* **1**, 59, (1993) (in Russian)

[7] V. V. Kassandrov, J. A. Rizcalla, gr-qc/9809056

[8] R. P. Kerr, W. B. Wilson *Gen. Rel. Grav.* **10**, 273, (1979)

[9] A. Ya. Burinskii, *in*: "Problems of theory of gravitation and elementary particles" **11**, 47, Moscow, Atomizdat, 1980 (in Russian)

[10] V. V. Kassandrov, J. A. Rizcalla, *in:* Abstracts of 10 Russian Gravitation Conference. Vladimir, 1999, p.122.

[11] B. Carter *Phys. Rev.* **174**, 1559, (1968)

[12] C. A. Lopez *Phys. Rev.* **D30**, 313, (1984)

[13] E. T. Newman *J. Math. Phys.* **14**, 102, (1973)

[14] W. Israel *Phys. Rev.* **D2**, 641, (1970)

[15] R. W. Lind, E. T. Newman *J. Math. Phys.* **15**, 1103, (1974)

[16] A. Ya. Burinskii, D. D. Ivanenko *Izvestiya Vuzov Fiz.* **5**, 135, (1975)

[17] A. Burinskii, G.Magli, gr-qc/9904012

[18] A. Burinskii, R. P. Kerr, Z.Perjés, gr-qc/9501012

[19] G. Bateman, "The Mathematical Analysis of Electrical and Optical Wave-Motion on the basis of Maxwell equations", Dover Publication Inc., 1955.

[20] G. Clement *Phys. Rev.* **D50**, R7119, (1994); gr-qc/9808082.

[21] A. F. Rañada, J. Trueba, *Phys.Lett.* **B422**, 196, (1998); hep-th/9802166

[22] V. N. Zhouravl'ev *in*: "Gravitation and Electromagnetism." No.6. Minsk, "Universitetskoe", 1998, p.105 (in Russian).